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A RIGID-PLASTIC ANALYSIS OF THE LIMITING EQUILIBRIUM OF AN EDGE CLEAVAGE CRACK IN A RECTANGULAR PLATE†

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The analytic solution of the plane rigid-plastic problem of an edge cleavage crack parallel to the base of a rectangular plate is constructed. The geometry of the domain and the loading conditions (concentrated loads acting along a normal to the crack line) simulate a widely used scheme of experiments for determining the crack resistance of materials. The solution of the problem leads to a scheme for the flow of the material ahead of the crack which is accompanied by the formulation of a transverse microcrack. A variational principle for problems of rigid-plastic analysis, taking into account formation of cracks, formulated and proved in [1], is used to solve the problem. Copyright © 1996 Elsevier Science Ltd.

1. FORMULATION OF THE PROBLEM

Consider the limiting state of a sample with a crack which has been subjected to a symmetric load (Fig. 1). We assume that the length of the sample L is long compared with its half-width H so that the plastic zones should reach the surface $y = \pm H$.

Let a plastic zone be formed ahead of the crack tip and let zones 2 and 3 be rigid. We shall obtain the boundary conditions on the boundary of the rigid and plastic zones. In the general case, a discontinuity in the tangential component of the velocity may occur at the boundary of the plastic and rigid zones and the jump in velocity is a constant quantity along the discontinuity line.

The boundary of zones 1 and 2 must intersect the axis of symmetry ($y = 0$) at an angle $\alpha = \pi/4$. Consequently, the velocity component which is normal to the boundary of zones 1 and 2 is not parallel to the y axis and the discontinuity in this component is equal to zero from the condition of the incompressibility of the material. In addition, the velocity component v_y at the intersection of the boundary of zones 1 and 2 with the axis of symmetry is equal to zero on both sides of the line of the assumed velocity discontinuity. As a consequence of this, it can be concluded that the jump in the velocity is equal to zero at the point of intersection of the boundary of zones 1 and 2 and the axis of symmetry $y = 0$. Consequently, there is no jump in the velocity at the boundary of zones 1 and 2 since, as was noted above, it preserves its magnitude along the whole of the discontinuity line.

We now construct a solution for the rigid-plastic boundary (line 1 in Fig. 2) and pick out a characteristic domain which is adjacent to the rigid-plastic boundary (the domain bounded by the lines 1, 2 and 3 in Fig. 2). In this domain, we have a mixed problem with the boundary conditions $v_x = 0$, $v_y = 0$ on line 1 which is a characteristic and $v_y = 0$ on line 2 which is not a characteristic. The boundary-value problem has the obvious solution $v_x = 0$, $v_y = 0$ in the whole of the domain under consideration and, by virtue of the uniqueness of the solution, the domain bounded by lines 1, 2 and 3 is rigid. Analogous reasoning can be continued until the point O_1 coincides with the crack tip, that is, with the point O. In addition, since the surface $y = H$ is plane and stress-free, the field of the characteristics which reaches this surface must be formed by families of straight lines which are inclined at angles of $\pm\pi/4$ to the axis (Fig. 2, the field of the characteristics close to the point O). If such a field of characteristics is obtained, fracture of the sample occurs by a rotation of the part of the sample 3 with respect to part 1 (Fig. 1) without the crack growing.

However, as the experimental data show, transverse cracks or crack-like defects frequently arise in the plastic zone in front of the crack tip. It is precisely this fact which is the basis of Cottrell's model [1], within the framework of which the fracture of an elastoplastic material and the occurrence of discontinuities in front of the main crack are attributed to kinematic causes. In the case of the rigid-plastic model being considered, the impossibility of plastic zones developing in front of the crack, as was shown above, is also associated with a kinematic process. It is therefore natural to assume in this case that transverse cracks develop, as in Cottrell's model, in front of the tip of the main crack.

There are therefore two possible cases.

1. Flow without a transverse crack and without a plastic zone.
2. Flow with the development of a transverse crack.

The choice of one or other velocity field can be made using a variational principle [2].

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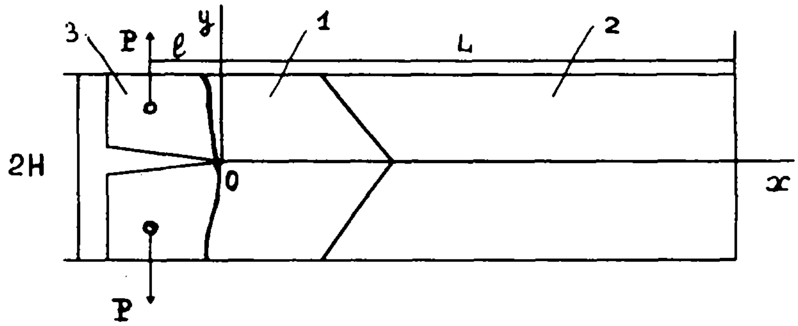

Fig. 1.

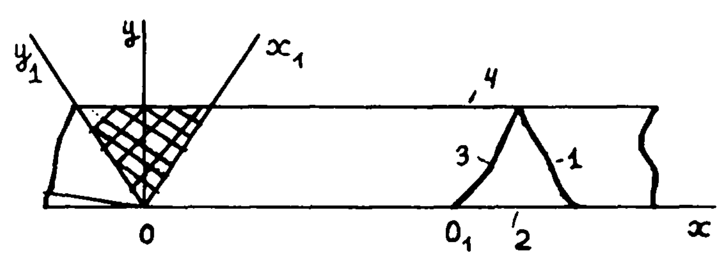

Fig. 2.

2. FLOW WITHOUT THE DEVELOPMENT OF TRANSVERSE CRACKS

In this case, the domain under consideration is localized close to the crack tip (Fig. 2). We introduce a new Cartesian system of coordinates x_1, y_1 which coincides with the characteristics bounding the plastic zone (Fig. 2). In the case of a rectilinear field of characteristics, the projections of the velocity v_{x1} and v_{y1} retain constant values along the corresponding characteristic lines [3]. When $y_1 = 0$, the projection $v_{y1} = 0$, since the material in zone 2 is stationary and the velocity component normal to the line $y_1 = 0$ must be continuous. Hence, $v_{y1} = 0$ on the whole of the plastic zone. Zone 3 rotates relative to the point O as an absolutely rigid body and the velocity v_{x1} on the line x_1 will therefore be

$$v_{x_1} = \omega y_1 \tag{2.1}$$

(ω is the angular velocity of zone 2). Since, in the case of a rectilinear field of characteristics, v_{x1} preserves its value along each of the characteristic lines parallel to x_1, expression (2.1) holds in the whole of the plastic zone.

Note that $v_{x1} = 0$ when $y_1 = 0$ and the velocities of the points of the rigid zone 2 which are adjacent to the plastic zone are perpendicular to the line $x_1 = 0$. There are therefore no discontinuities in the velocities along the lines $x_1 = 0$ and $y_1 = 0$. The deformation rates from (2.1) and the conditions $v_{y1} = 0$ are determined as

$$\varepsilon_{x_1} = 0, \quad \varepsilon_{y_1} = 0, \quad \varepsilon_{x_1 y_1} = \omega / 2 \tag{2.2}$$

The power of the plastic work, when the Mises yield condition is used, is determined by the expression

$$q = K(2\varepsilon_{ij}\varepsilon_{ij})^{1/2} \tag{2.3}$$

Here K is the yield point in the case of pure shear. For the case under consideration, from (2.2) and (2.3) we find

$$q_1 = K\omega \tag{2.4}$$

whence the classical method of an upper estimate gives

$$P_1 l \omega = \int_\Omega q_1 d\Omega \tag{2.5}$$

Here P_1 is an estimate of P for the velocity field under consideration and ω is the area of the plastic zone.

From (2.4) and (2.5), we find

$$P_1 = KH^2/l \tag{2.6}$$

3. FLOW WITH THE DEVELOPMENT OF A TRANSVERSE CRACK

We now consider flow assuming that a transverse crack develops in front of the main crack. In accordance with the definition given in [1], the kinematically possible velocity field can be taken in the form shown in Fig. 3, where 2 is a rigid zone which is analogous to zone 2 in Fig. 1, 1 and 3 are plastic zones formed by rectilinear families of characteristics, and 4 is a transverse crack.

In this case the rigid zone 2 rotates about the point O_2. Expression (2.4) for the power of the plastic work holds good in the plastic domain 1. In the plastic domain 3, we introduce the Cartesian system of coordinates x_2y_2 as shown in Fig. 3. When $y_2 = 0$, we obtain

$$v_{y_2} = -\omega x_2 \tag{3.1}$$

from the condition of continuity of the normal component of the velocity.

In the system of coordinates x_2y_2, the axis of symmetry is defined by the equation

$$x_2 + y_2 = \sqrt{2}h \tag{3.2}$$

The condition

$$v_y = v_{x_2}\cos(\pi/4) + v_{y_2}\cos(\pi/4) = 0$$

must be satisfied on this axis.

From this, on the line (3.2), we have

$$v_{x_2} = -v_{y_2} \tag{3.3}$$

Substituting (3.1) and (3.2) into (3.3), we obtain

$$v_{x_2} = -\omega(y_2 - \sqrt{2}h) \tag{3.4}$$

and, since $v_{x2} = \sqrt{(2)}\omega h$ when $y_2 = 0$, the discontinuity in the tangential component of the velocity along the x_2 axis will be

$$[v_{x_2}] = \sqrt{2}\omega h \tag{3.5}$$

From (3.1) and (3.4), we find

$$\varepsilon_{x_2} = 0, \quad \varepsilon_{y_2} = 0, \quad \varepsilon_{x_2y_2} = -\omega$$

Then, from (2.3)

$$q_3 = 2K\omega \tag{3.6}$$

The power of the plastic work in the whole of the domain will be

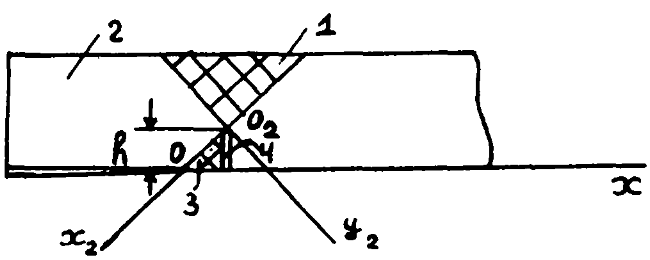

Fig. 3.

$$Q_2 = \int_{\Omega_1} q_1 d\Omega + \int_{\Omega_3} q_3 d\Omega + K[v_{x_2}]\sqrt{2}\,h \tag{3.7}$$

Here, Ω_1 is the area of zone 1 and Ω_3 is the area of zone 3.

Substituting (2.4), (3.5) and (3.6) into (3.7), we obtain

$$Q_2 = K\omega[(H-h)^2 + 3h^2] \tag{3.8}$$

The discontinuity in the normal component of the velocity on line 4 is determined by the equality $[v_n] = \omega h$. Then, the power which is required to form a finite crack will be

$$Q_c = \sigma_0 \omega h^2 \tag{3.9}$$

(σ_0 is a constant of the material). From (3.8) and (3.9), we obtain the balance of the powers of the external and internal forces

$$P(l+h)\,\omega = K\omega[(H-h)^2 + 3h^2] + \sigma_0 \omega h^2 \tag{3.10}$$

It follows from the variational principle that an estimate of the load P_2 for the given velocity field can be obtained from the condition for a minimum of P as a function of h.

We now introduce the dimensionless quantities $l' = l/H$, $h' = h/H$, $\gamma = \sigma_0/K$ and from here on we shall omit the primes. Then, by (3.10), from the equation $dP/dh = 0$, subject to the condition that $h \geqslant 0$, we find

$$h = -l + \lambda, \quad \lambda = [l^2 + (1+2l)/(4+\gamma)]^{1/2} \tag{3.11}$$

An estimate of the loading is obtained from (3.10) and (3.11). In dimensionless form, we have

$$\bar{P}_2 = P_2/(KH) = 2\{[1 + 2l + l^2(4+\gamma)]/\lambda - [1 + l(4+\gamma)]\} \tag{3.12}$$

4. ANALYSIS OF THE SOLUTIONS AND CONCLUSIONS

We determine the ratio of the limiting loads from (2.6) and (3.12).

The dependence of the ratio of the limiting loads ξ (the solid curves) and the dimensionless length of the crack h (the dashed curves) on l for different values of γ is shown in Fig. 4. It is clear that the velocity field with the transverse crack gives a smaller value of the limiting load for all positions of the point of application of the load for which calculations have been carried out. Consequently, a solution with a crack more correctly reflects the actual pattern of the flow of a material during the fracture of a sample.

It is likely that a better estimate of the limiting load could be found if another velocity field with a crack were to be chosen. However, it appears to be impossible to find a kinematically admissible velocity field without a crack which would improve the estimate (2.6). It follows that, in experiments of the type being considered, transverse cracks will develop in front of the main crack if the deformation of the material can be described using a rigid-plastic model.

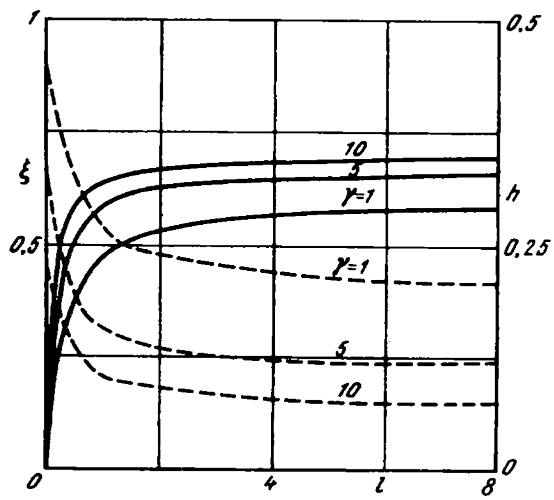

Fig. 4.

It should be noted that such behaviour of a material is typical for the type of experiments being considered. If, for example, one considers the bending or stretching of a sample with cracks, then the previous conclusions concerning the impossibility of plastic zones in front of a crack will be untrue since, in such cases, contact between the rigid and deformable zones on the axis of symmetry is only possible at a single point. The condition that the velocity normal to the axis of symmetry of the rigid zone should be equal to zero, which holds, for example, in the case of Prandtl's problem on the compression of a layer between rough sheets [3], therefore cannot be satisfied. A similar conclusion can be drawn concerning the sample under consideration if the magnitude of L is sufficiently small.

It is obvious that the previous conclusions hold if one is not considering a crack but a symmetric notch of any shape. Furthermore, it is possible to seek a solution with several transverse cracks which develop along the axis of symmetry.

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REFERENCES

1. COTTRELL A. H., Mechanics of fracture in large structures. *Proc. Roy. Soc. Ser. A* **285**, 1400, 10–21, 1965.
2. ALEKSANDROV S. E. and GOL'DSHTEIN R. V., The variational principle in the theory of rigid-plastic flow taking the formation of cleavage cracks into account. *Dokl. Ross. Akad. Nauk* **342**, 4, 484–486, 1995.
3. HILL R., *The Mathematical Theory of Plasticity*. Clarendon Press, Oxford, 355, 1983.

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